NEW COMPARISON THEOREMS IN RIEMANNIAN GEOMETRY

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ABSTRACT. We construct and use solutions, subsolutions, and supersolutions of differential equations as catalysts to link the hypotheses on radial curvature of a complete $n$-manifold $M$ to the conclusions on the analysis or geometry of quadratic forms and second order differential operators. In particular, we prove Hessian Comparison Theorems and Laplacian Comparison Theorems on $M$, generalizing the work of Greene and Wu: If the radial curvature $K$ of $M$ satisfies

$$-\frac{a^2}{c^2 + r^2} \leq K(r) \leq \frac{b^2}{c^2 + r^2}$$

on $D(x_0)$ where $0 \leq a^2, 0 \leq b^2 \leq \frac{1}{4}$, $0 \leq c^2$, and $D(x_0) = M \setminus \text{Cut}(x_0)$, then

$$\frac{1 + \sqrt{1 - 4b^2}}{2r} \left( g - dr \otimes dr \right) \leq \text{Hess}(r) \leq \frac{1 + \sqrt{1 + 4a^2}}{2r} \left( g - dr \otimes dr \right)$$

on $D(x_0)$, in the sense of quadratic forms, and

$$(n - 1) \frac{1 + \sqrt{1 - 4b^2}}{2r} \leq \Delta r \leq (n - 1) \frac{1 + \sqrt{1 + 4a^2}}{2r}$$

holds pointwise on $D(x_0)$, and $\Delta r \leq (n - 1) \frac{1 + \sqrt{1 + 4a^2}}{2r}$ holds weakly on $M$. A volume comparison theorem is also given.

This is based on my joint research work with Yingbo Han, Yibin Ren and Shihshu Walter Wei, and further work with Shihshu Walter Wei.